# A certain class of standard subalgebras of affine Kac-Moody algebras

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**Abstract**

The aim of this paper is to extend the theory of standard subalgebras of finite dimensional simple Lie algebras to infinite dimensional Lie algebras. We construct and characterize a class of standard subalgebras of affine Kac-Moody algebra.

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# Introduction

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$, of rank $p$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by $\Delta$ the roots system of the couple $(\mathfrak{g}, \mathfrak{h})$ and by $\Pi = \{\alpha_1, \cdots, \alpha_p\}$ the simple roots of $\Delta$.

Let $A$ be a generalized Cartan matrix of affine type of order $p+1$ associated to finite dimensional simple Lie algebra $\mathfrak{g}$.

Let $S(A)$ (resp. $S({}^tA)$) be the Dynkin diagram associated to $A$ (resp. ${}^tA$) and $\mathfrak{B} = \{\alpha_0, \cdots, \alpha_p\}$ (resp. $\mathfrak{B}^\vee = \{\alpha_0^\vee, \cdots, \alpha_p^\vee\}$) be the set of the vertex of diagram $S(A)$ (resp. $S({}^tA)$) called roots basis (resp. coroots basis). Let $a_0, a_1, \cdots, a_p$ be the numerical labels of $S(A)$ and $a_0^\vee, \cdots, a_p^\vee$ be the numerical labels of $S({}^tA)$.

We denote by $\mathcal{L} := \mathbb{C}[t, t^{-1}]$ the algebra of Laurent polynomials in $t$. Recall that the residue of a Laurent polynomial $P = \sum_{i \in \mathbb{Z}} c_i t^i$ (where all but a finite number of $c_i$ are 0) is defined by $ResP = c_{-1}$.

Consider the infinite dimensional Lie algebra, called *the Loop algebra*

$$\mathcal{L}(\mathfrak{g}) := \mathcal{L} \otimes \mathfrak{g}$$

We denote by $[.,.]$ the bracket on $\mathfrak{g}$. The bracket on $\mathcal{L}(\mathfrak{g})$ is defined as follows, for all $(x, y) \in \mathfrak{g} \times \mathfrak{g}$ : $[t^n \otimes x, t^m \otimes y]_{\mathcal{L}} = t^{n+m} \otimes [x, y]$.

Fix a nondegenerate invariant symmetric bilinear $\mathbb{C}$-valued form $(./.)$ on $\mathfrak{g}$. We extend this form by linearity to an $\mathcal{L}$-valued bilinear form $(./.)_L$ on $\mathcal{L}(\mathfrak{g})$ which is defined by

$$(P \otimes x/Q \otimes y)_L = PQ(x/y)$$

Now, we define a $\mathbb{C}$-valued 2-cocycle on the Lie algebra $\mathcal{L}(\mathfrak{g})$ by

$$\psi(a,b) = Res(\frac{da}{dt}, b)_t \qquad \text{for all } (a,b) \in \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g})$$

We denote by $\tilde{\mathcal{L}}(\mathfrak{g})$ the extension of the Lie algebra $\mathcal{L}(\mathfrak{g})$ by 1-dimensional center, associated to the cocycle $\psi$. Namely, $\tilde{\mathcal{L}}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}\mathrm{K}$ where $\mathrm{K} = \sum_{i=0}^{i=p} a_i^{\vee}\alpha_i^{\vee}$ is a generator of the 1-dimensional center.

Finally, the affine Kac-Moody algebra associated to matrix $A$ is

$$\mathfrak{g}(A) := \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}\mathrm{K} \oplus \mathbb{C}\mathtt{d}$$

where $\mathtt{d}$ is the derivation defined by : $\mathtt{d} = t\frac{d}{dt}$ on $\mathcal{L}(\mathfrak{g})$, $\mathtt{d}(\mathrm{K}) = 0$. For more details of this construction see [8, chap. 7].

The aim of this paper is to construct a certain class of standard subalgebras of affine Kac-Moody algebras $\mathfrak{g}(A)$. This uses standard subalgebras of the complex simple Lie algebra $\mathfrak{g}$.

Let $\tau$ be a subalgebra of $\mathfrak{g}$. *If the normalizer of $\tau$ is a parabolic subalgebra of $\mathfrak{g}$ then $\tau$ is called standard subalgebra of $\mathfrak{g}$.* Several papers were devoted to the study of standard subalgebras for finite dimensional case, see [6], [9], [13],[10], [11].

These subalgebras were characterized in [9] and [13] using a noncomparable roots with respect to the partial order relation " $\leqslant$ " defined on the dual vector space $\mathfrak{h}^*$ by $\omega_1 \leqslant \omega_2$ if $\omega_2 - \omega_1$ is linear combination of simple roots of $\Pi$ with non-negative coefficients.

Let $\mathfrak{R}$ be a set of positive roots two by two noncomparable. Consider the set $\mathfrak{R}_1 = \{\beta \in \Delta_+ : \omega \leqslant \beta$ for certain $\omega \in \mathfrak{R}\}$. Then the subalgebra $\mathfrak{m} = \sum_{\beta \in \mathfrak{R}_1} \mathfrak{g}_\beta$ is nilpotent standard subalgebra of $\mathfrak{g}$.

Setting $\rho(\mathfrak{m})$ be the normalizer of $\mathfrak{m}$. Let $\mathfrak{r}$ be an ideal of the Levi reductive subalgebra $\mathfrak{h} + \sum_{\alpha \in \Omega_1} \mathfrak{g}_\alpha$, lying in the Levi reductive subalgebra associated to parabolic subalgebra $\rho(\mathfrak{m})$ where $\Omega_1$ is the set of roots which are linear combination of $\Pi \backslash (\Pi \cap \mathfrak{R})$.

Any standard subalgebra $\tau$ of $\mathfrak{g}$ has the form

$$\tau = \mathfrak{m} + \mathfrak{r}$$

and the normalizer $\rho(\tau) = \rho(\mathfrak{m})$.

In this paper, theorem 1.1 states that there exists standard subalgebra $\overline{\tau}$ of $\mathfrak{g}(A)$ associated to a given standard subalgebra $\tau$ of finite dimensional simple Lie algebra $\mathfrak{g}$ and a certain vector subspace $V_\tau = [\tau, \mathfrak{g}]$ of $\mathfrak{g}$ associated to $\tau$. The standard subalgebra $\overline{\tau}$ has the form

$$\overline{\tau} = t^n \otimes \tau + t^{n+1} \otimes V_\tau + t^{n+2}\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}K \quad \text{ for } n \in \mathbb{N}^*$$

The subspace $V_\tau$ is called "appui" subspace of $\mathfrak{g}$.

The standard subalgebra $\overline{\tau}$ is determined by the finite dimensional standard subalgebra $\tau$ and the appui subspace $V_\tau$. Explicit formulas of $V_\tau$ are established for different cases of $\tau$.

In the case where $\tau$ is nilpotent standard subalgebra. Then, $\tau$ is associated to subset $\mathfrak{R} \subset \Delta_+$ of positive roots two by two noncomparable. The theorem 2.1 states that the appui subspace $V_\tau$ has the following form :

$$V_\tau = \sum_{\alpha \in \Delta_+ \backslash <\mathfrak{R}_3>^+} \mathfrak{g}_\alpha + \sum_{\alpha \in \mathfrak{R}_1 \cup \mathfrak{R}_2} \mathbb{C}[e_\alpha, e_{-\alpha}] + \sum_{\alpha \in \mathfrak{R}_2} \mathfrak{g}_{-\alpha}$$

where $\mathfrak{R}_1$, $\mathfrak{R}_2$ and $\mathfrak{R}_3$ are defined by the relation 1, 3 and 4.

In the second case where $\tau$ is standard subalgebra of nilpotent radical. Then, $\tau = \mathfrak{m} + \mathfrak{r}_0$ where $\mathfrak{m}$ is nilpotent part of $\tau$ and $\mathfrak{r}_0$ is semisimple part of $\tau$. Therefor, $\mathfrak{r}_0$ is constructed from a certain common connected subsystem $\psi$ of $\Pi \backslash (\Pi \cap \mathfrak{R})$ and $\Pi \backslash (\underset{\omega \in \mathfrak{R}}{\cup} S^\omega)$ where $S^\omega = \{\gamma \in \Pi \ : \ \omega = \gamma \text{ or } \omega - \gamma \in \Delta\}$. One may write $\mathfrak{r}_o = \sum_{\alpha \in \langle \Psi \rangle} \mathfrak{g}_\alpha + \sum_{\alpha \in \langle \Psi \rangle} \mathbb{C}[e_\alpha, e_{-\alpha}]$ where $\langle \psi \rangle$ is the set of roots which are linear combination of elements of $\psi$. We prove in theorem 2.2 that the appui subspace $V_\tau$ has the following form :

$$\begin{array}{l} V_\tau = V_\mathfrak{m} \quad \text{if} \quad P_\Psi^- = \varnothing \\ V_\tau = \mathfrak{g} \quad \text{if} \quad P_\Psi^- \neq \varnothing \end{array}$$

where $P_\Psi^- = \sum_{\alpha \in \langle \Psi \rangle} \sum_{\beta \in \Delta_+ \backslash \mathfrak{R}_2} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\beta}]$ and $V_\mathfrak{m}$ is the appui subspace associated to $\mathfrak{m}$.

# 1 Standard subalgebras of affine Kac-Moody algebra

Let $\mathfrak{g}$ be a simple Lie algebra of rank $p$. Let $A$ be its extended Cartan matrix of affine type and $\mathfrak{g}(A)$ be the affine Kac-Moody algebra associated to $A$. We consider the element $\delta$ defined by

$$\delta = \sum_{i=0}^{i=p} a_i \alpha_i$$

Let $\Delta^{aff} = \Delta^{re} \cup \Delta^{im}$ be a roots system of the couple $(\mathfrak{g}(A), \mathfrak{h} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathsf{d})$ where $\Delta^{re} = \{\alpha + j\delta \ : \ \alpha \in \Delta \ \text{ and } \ j \in \mathbb{Z}\}$ is the set of real roots and $\Delta^{im} = \{j\delta \ : \ j \in \mathbb{Z}\}$ is the set of imaginary roots.

We denote by $[.,.]$ the bracket on $\mathfrak{g}$. The bracket $[.,.]_{aff}$ on $\mathfrak{g}(A)$ is defined as follows : for all $(x,y) \in \mathfrak{g} \times \mathfrak{g}$ , $(\lambda, \mu, \lambda_1, \mu_1) \in \mathbb{C}^4$ , $(n,m) \in \mathbb{Z}^2$ :

$$[t^n \otimes x + \lambda \mathrm{K} + \mu \mathsf{d}, t^m \otimes y + \lambda_1 \mathrm{K} + \mu_1 \mathsf{d}]_{aff} = t^{n+m} \otimes [x,y] + \mu m t^m \otimes y - \mu_1 n t^n \otimes x + n\delta_{n,-m}(x|y)\mathrm{K}$$

In the particular case where $m$ and $n$ are two natural numbers, we have :

$$[t^n \otimes x + \lambda \mathrm{K} + \mu \mathsf{d}, t^m \otimes y + \lambda_1 \mathrm{K} + \mu_1 \mathsf{d}]_{aff} = t^{n+m} \otimes [x,y] + \mu m t^m \otimes y - \mu_1 n t^n \otimes x$$

**Definition 1** *We introduce the following definitions :*

1. *A subalgebra of $\mathfrak{g}(A)$ is called parabolic if it contains a Borel subalgebra of $\mathfrak{g}(A)$.*

2. *A subalgebra of $\mathfrak{g}(A)$ is called standard if its normalizer is a parabolic subalgebra of $\mathfrak{g}(A)$.*

Let $\tau$ be a standard subalgebra of $\mathfrak{g}$ of normalizer $\rho(\tau)$ and let $V_\tau = [\tau, \mathfrak{g}]$ be the associated subspace.

**Proposition 1.1** *The appui subspace $V_\tau$ satisfies the following relation : $[V_\tau, \rho(\tau)] \subset V_\tau$.*

**Proof :** Using the Jacobi identity, we have $[V_\tau, \rho(\tau)] = [[\tau, \mathfrak{g}], \rho(\tau)] = [\tau, [\mathfrak{g}, \rho(\tau)]] + [[\tau, \rho(\tau)], \mathfrak{g}] \subset [\tau, \mathfrak{g}] + [\tau, \mathfrak{g}] \subset [\tau, \mathfrak{g}] = V_\tau$. ∎

The following main theorem gives a sufficient condition for the existence of a class of standard subalgebras of $\mathfrak{g}(A)$.

**Theorem 1.1** *Let $\tau$ be a standard subalgebra of $\mathfrak{g}$ of normalizer $\rho(\tau)$ and $V_\tau = [\tau, \mathfrak{g}]$ be the appui subspace associated to $\tau$.*

*Then, the subalgebra*

$$\overline{\tau} = t^n \otimes \tau + t^{n+1} \otimes V_\tau + t^{n+2}\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}K \quad (n \in \mathbb{N}^*)$$

*is a standard subalgebra of $\mathfrak{g}(A)$ with normalizer $\rho(\overline{\tau}) = \rho(\tau) + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}K + \mathbb{C}d$.*

**Proof :** We show that the parabolic subalgebra $\rho(\overline{\tau})$ is the normalizer of $\overline{\tau}$.

In $\mathfrak{g}(A)$, the subalgebra $\overline{\tau}$ is an ideal of parabolic subalgebra $\rho(\tau) + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathtt{d}$.

Let $\rho_1$ be a parabolic subalgebra of $\mathfrak{g}$ such that $\rho_1 + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathtt{d}$ is a normalizer of $\overline{\tau}$. We have $[\overline{\tau}, \rho_1 + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathtt{d}]_{aff} \subset \overline{\tau}$ then $[\tau, \rho_1] \subset \tau$ and $\rho_1 \subset \rho(\tau)$. Since $\rho_1 + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathtt{d}$ is a normalizer of $\overline{\tau}$ then, we have $\rho_1 + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathtt{d}$ included in $\rho(\tau) + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathtt{d}$ and $\rho(\tau)$ included in $\rho_1$. Furthermore, $\rho_1 = \rho(\tau)$ and $\rho(\tau) + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathtt{d}$ is a normalizer of $\overline{\tau}$.
∎

**Corollary 1.1** *The subalgebra*

$$\tau + t \otimes \mathfrak{g} + t^2\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}K + \mathbb{C}d$$

*is a standard subalgebra of $\mathfrak{g}(A)$ of normalizer $\rho(\tau) + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}K + \mathbb{C}d$.*

**Corollary 1.2** *Let $V$ be a subspace of $\mathfrak{g}$ such that $V_\tau \subset V$ and $[V, \rho(\tau)] \subset V$ then the subalgebra*

$$t^n \otimes \tau + t^{n+1} \otimes V + t^{n+2}\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}K \quad (n \in \mathbb{N}^*)$$

*is standard subalgebra of $\mathfrak{g}(A)$.*

**Remark 1.1** *We deduce from the theorem 1.1 that the subalgebra* $\overline{V}_\tau = t^{n+1} \otimes V_\tau + t^{n+2}\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}K$ *is also standard subalgebra of* $\mathfrak{g}(A)$.

**Eexample 1.1** *Let* $\mathfrak{g}$ *be a simple Lie algebra of type* $B_4$ *and* $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ *be a basis of roots system* $\Delta$ .
*We set* $\mathfrak{R}_1 = \{\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4\}$. *We consider* $\tau = \sum\limits_{\alpha\in\mathfrak{R}_1} \mathfrak{g}_\alpha$ *and* $V_\tau = \sum\limits_{\alpha\in\Delta_+\setminus\{\alpha_4\}} \mathfrak{g}_\alpha + \mathfrak{h} + \mathfrak{g}_{-\alpha_1} + \mathfrak{g}_{-\alpha_2} + \mathfrak{g}_{-\alpha_1-\alpha_2}$ *respectively the standard subalgebra of* $\mathfrak{g}$ *and the appui subspace associated to* $\tau$.

*Then* $\overline{\tau} = t \otimes \tau + t^2 \otimes V + t^3\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K}$ *is a standard subalgebra of* $\mathfrak{g}(A)$ *with normalizer* $\rho(\overline{\tau}) = (\sum\limits_{\alpha\in\Delta_+} \mathfrak{g}_\alpha + \mathfrak{h} + \mathfrak{g}_{-\alpha_1} + \mathfrak{g}_{-\alpha_2} + \mathfrak{g}_{-\alpha_1-\alpha_2} + \mathfrak{g}_{-\alpha_4}) + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}d$.

*The standard subalgebra* $\overline{\tau}$ *may be written using the real and imaginary roots as follows :*

$$\overline{\tau} = \sum_{\alpha\in\mathfrak{R}_1} \mathfrak{g}(A)_{\alpha+\delta} + \sum_{\alpha\in\mathfrak{R}_\tau} \mathfrak{g}(A)_{\alpha+2\delta} + \mathfrak{g}(A)_{2\delta} + \sum_{n\geqslant 3}\{\sum_{\alpha\in\Delta} \mathfrak{g}(A)_{\alpha+n\delta} + \mathfrak{g}(A)_{n\delta}\} + \mathbb{C}\mathrm{K}$$

*with* $\mathfrak{R}_\tau = \Delta_+\setminus\{\alpha_4\}\cup\{-\alpha_1, -\alpha_2, -\alpha_1-\alpha_2\}$, $\mathfrak{g}(A)_{\alpha+n\delta} = t^n \otimes \mathfrak{g}_\alpha$ *and* $\mathfrak{g}(A)_{n\delta} = t^n \otimes \mathfrak{h}$

Theorem 1.1 proves that the existence of this class depends on the description of the standard subalgebras $\tau$ and the appui subspaces $V_\tau$ of the Lie algebra $\mathfrak{g}$.

In the next section, we characterize the appui subspaces $V_\tau$ when $\tau$ is the nilpotent standard subalgebra then when $\tau$ is the standard subalgebra.

# 2 Appui subspaces

Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{h}$ be its Cartan subalgebra and $\Delta$ be the roots system of the couple $(\mathfrak{g}, \mathfrak{h})$. Let $\tau$ be the standard subalgebra of $\mathfrak{g}$. The aim of this section is to give an explicit formulas for the appui subspaces $V_\tau = [\tau, \mathfrak{g}]$.

## 2.1 Case of nilpotent standard subalgebra

**Main notations.**
1. For any positive root $\beta$, we denote by $S^\beta$ the set of simple roots $\gamma$ of $\Pi$ such that $\beta = \gamma$ or $\beta - \gamma \in \Delta$. The set $S^\beta$ is called set of extremal roots of $\beta$.
2. For any root $\alpha = \sum\limits_{\gamma\in\Pi} \alpha_\gamma\gamma$, we denote by $C_\alpha$ the set of the simple roots $\gamma$ in $\Pi$ such that $\alpha_\gamma \neq 0$.
3. Let $B$ be the subset of the simple roots $\Pi$. We denote by $\langle B\rangle$ the set of roots which are linear combination of elements of $B$ and by $\langle B\rangle^+$ (resp. $\langle B\rangle^-$) the positive roots (resp. the negative roots) of $\langle B\rangle$.

Let $\tau$ be a nilpotent standard subalgebra of $\mathfrak{g}$ associated to the subsystem $\mathfrak{R}$ of positive roots two by two non comparable. We set

$$\mathfrak{R}_1 = \{\, \beta \in \Delta_+ : \quad \omega \leqslant \beta \quad \text{for a certain } \omega \in \mathfrak{R}\} \tag{1}$$

The subalgebra $\tau$ has the form $\tau = \sum\limits_{\beta\in\mathfrak{R}_1} \mathfrak{g}_\beta$.

We set $S_2 = \bigcup_{\omega\in\mathfrak{R}} S^\omega$, $\Delta_1 = \langle \Pi\backslash S_2\rangle$ and $\Delta_2 = \Delta\backslash\Delta_1$. The normalizer of $\tau$ is defined by $S_2$ and has the following form $\rho(\tau) = \sum_{\alpha\in\Delta_+}\mathfrak{g}_\alpha + \mathfrak{h} + \sum_{\alpha\in\Delta_1^+}\mathfrak{g}_{-\alpha}$.

We can write $\mathfrak{g} = \rho(\tau) + \sum_{\nu\in\Delta_2^+}\mathfrak{g}_{-\nu}$. Then, we have

$$V_\tau = [\tau,\mathfrak{g}] \subset \tau + \sum_{\beta\in\mathfrak{R}_1}\sum_{\nu\in\Delta_2^+}[\mathfrak{g}_\beta,\mathfrak{g}_{-\nu}]. \tag{2}$$

**Lemma 2.1** *Let $\alpha$ be a positive root. Then $\mathfrak{g}_\alpha$ is included in $V_\tau$ if and only if there exists $\beta\in\mathfrak{R}_1$ such that $C_\alpha\cap S^\beta\neq\varnothing$.*

**Proof :** ($\Longrightarrow$) Let $\alpha\in\Delta_+$ such that $\mathfrak{g}_\alpha\subset V_\tau$. There are two cases :
First case : $\alpha\in\mathfrak{R}_1$ then there exists $\beta\in\mathfrak{R}$ such that $\beta\leqslant\alpha$. This implies that $C_\alpha\cap S^\beta\neq\varnothing$.
Second case : $\alpha\notin\mathfrak{R}_1$, using the formula 2, there exists a root $\beta\in\mathfrak{R}_1$ such that $\beta-\alpha$ is a root of $\Delta_2^+$. One we can write $\alpha$ in the form $\alpha=\alpha_1+\cdots+\alpha_k$ with $\alpha_1,\cdots,\alpha_k$ simple roots and each partial sum $\alpha_1+\cdots+\alpha_l$ is a root, for $1\leqslant l\leqslant k$. We want to prove by induction on $k$ that there exists $i_0\in[\![1,k]\!]$ such that $\alpha_{i_0}$ is in $S^\beta$.

If $k=1$. We have $\beta-\alpha_1\in\Delta$. Then $\alpha_1\in S^\beta$. Suppose that $k>1$, then we have $-\alpha'_{k-1}-\alpha_k+\beta$ is a root where $\alpha'_{k-1}=\alpha_1+\cdots+\alpha_{k-1}$. By applying Jacobi identity to the roots $-\alpha'_{k-1},-\alpha_k$ and $\beta$, we have $\beta-\alpha'_{k-1}$ or $\beta-\alpha_k$ is a root.
If $\beta-\alpha'_{k-1}\in\Delta$, by the assumption of recurrence, there exists $j_0\in[\![1,k-1]\!]$ such that $\alpha_{j_0}\in S^\beta$ and $C_\alpha\cap S^\beta\neq\varnothing$.
If $\beta-\alpha_k\in\Delta$, we have $\alpha_k\in S^\beta$ and $C_\alpha\cap S^\beta\neq\varnothing$.

($\Longleftarrow$) Let $\alpha$ be a root of $\Delta_+$ such that there exists $\beta$ in $\mathfrak{R}_1$ with $C_\alpha\cap S^\beta\neq\varnothing$. We set $\alpha=\alpha_1+\cdots+\alpha_n$ such that for all $i\in[\![1,n]\!]$, $\alpha_i\in\Pi$ and for $r\in[\![1,n]\!]$ each partial sum $\alpha_1+\cdots+\alpha_r$ is a root.

Let $J$ be a subset of $[\![1,n]\!]$ such that $\alpha_j\in C_\alpha\cap S^\beta$ for any $j\in J$. It is clear that the subset $J$ is not empty. Let $q$ be an element of $J$. We have $\beta-\alpha_q$ is a root and $\mathfrak{g}_{\alpha_q}=[\mathfrak{g}_\beta,\mathfrak{g}_{-\beta+\alpha_q}]\subset[\tau,\mathfrak{g}]=V_\tau$.

Using the proposition 1.1, we may deduce that

$$\mathfrak{g}_\alpha=\mathfrak{g}_{\alpha_1+\cdots+\alpha_n}=[[\mathfrak{g}_{\alpha_q},\mathfrak{g}_{\alpha_1+\cdots+\alpha_{q-1}}],\mathfrak{g}_{\alpha_{q+1}},\cdots,\mathfrak{g}_{\alpha_n}]\subset[V_\tau,\rho(\tau)]\subset V_\tau$$

■

**Lemma 2.2** *Let $\alpha$ be a positive root. Then $\mathfrak{g}_{-\alpha}$ is included in $V_\tau$ if and only if there exists $\beta\in\mathfrak{R}_1$ such that $\alpha+\beta\in\Delta_+$.*

**Proof :** ($\Longrightarrow$) Let $\alpha\in\Delta_+$ such that $\mathfrak{g}_{-\alpha}\subset V_\tau$. Then, by relation 2, there exists a root $\beta\in\mathfrak{R}_1$ such that $\alpha+\beta$ is a root of $\Delta_2^+$.

($\Longleftarrow$) Let $\beta$ be a root in $\mathfrak{R}_1$ such that $\alpha+\beta$ is a root. Then we have $\mathfrak{g}_{-\alpha}=[\mathfrak{g}_\beta,\mathfrak{g}_{-\beta-\alpha}]\subset[\tau,\mathfrak{g}]=V_\tau$. ■

We consider the set $\mathfrak{R}_2$ of $\Delta$ satisfying the following property :

$$\text{If } \alpha\in\Delta_+,\beta\in\mathfrak{R}_1, \text{ and } \alpha+\beta\in\Delta \text{ then } \alpha\in\mathfrak{R}_2 \tag{3}$$

And the set $\mathfrak{R}_3$ of $\Pi$ satisfying the following property :

$$\text{If } \alpha \in \Pi, \beta \in \mathfrak{R}_1, \text{ and } \beta - \alpha \notin \Delta \text{ then } \alpha \in \mathfrak{R}_3 \tag{4}$$

In other words $\mathfrak{R}_3 = \Pi \backslash S_2$

The next theorem characterizes the appui subspace $V_\tau$ when $\tau$ is a nilpotent standard subalgebra of $\mathfrak{g}$.

**Theorem 2.1** *Let $\mathfrak{g}$ be a complex simple Lie algebra and $\tau$ be a nilpotent standard subalgebra associated to a subsystem $\mathfrak{R}$ of positive roots two by two non comparable. Then, the appui subspace associated to $\tau$ has the form*

$$V_\tau = \sum_{\alpha \in \Delta_+ \backslash <\mathfrak{R}_3>^+} \mathfrak{g}_\alpha + \sum_{\alpha \in \mathfrak{R}_1 \cup \mathfrak{R}_2} \mathbb{C}[e_\alpha, e_{-\alpha}] + \sum_{\alpha \in \mathfrak{R}_2} \mathfrak{g}_{-\alpha}$$

**Proof :** By applying the lemma 2.1 and 2.2, it is enough to prove that $\sum\limits_{\alpha \in \mathfrak{R}_1 \cup \mathfrak{R}_2} \mathbb{C}[e_\alpha, e_{-\alpha}]$ is included in $V_\tau$.
If $\alpha \in \mathfrak{R}_1$ then $\mathbb{C}[e_\alpha, e_{-\alpha}] \subset [\tau, \mathfrak{g}] = V_\tau$. If $\alpha \in \mathfrak{R}_2$ then, we have $\mathfrak{g}_{-\alpha}$ is included in $V_\tau$ and by applying the proposition 1.1, we have $\mathbb{C}[e_{-\alpha}, e_\alpha] \subset [V_\tau, \rho(\tau)] \subset V_\tau$. ∎

**Remark 2.1** *We denote by $\theta$ the highest root of $\Delta$. If the $\mathfrak{R}$is included in basis $\Pi$, the nilpotent standard subalgebra $\tau$ is called complete standard subalgebra of $\mathfrak{g}$ and it is easy to prove that the appui subspace $V_\tau$ has the form*

$$V_\tau = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha + \mathfrak{h} + \sum_{\alpha \in \Delta_+ \backslash \mathfrak{R}_C} \mathfrak{g}_{-\alpha}$$

*with $\mathfrak{R}_C = \{\alpha \in \Delta_+ :$ for all root $\beta \in \mathfrak{R}$ , $\alpha_\beta = \theta_\beta\}$.*

**Remark 2.2** *We consider $m = \sum\limits_{\alpha \in \Delta} \mathfrak{g}_\alpha$ the nilpotent standard subalgebra associated to the basis $\Pi$. It is the maximal ad-nilpotent ideal of Borel subalgebra of $\mathfrak{g}$. Then, the appui subspace is*

$$V_\tau = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha + \mathfrak{h} + \sum_{\alpha \in \Delta_+ \backslash \{\theta\}} \mathfrak{g}_{-\alpha}$$

## 2.2 General case

In this section, we establish an explicit formula for the appui subspaces $V_\tau$ when $\tau$ is a standard subalgebra of $\mathfrak{g}$ not necessary nilpotent.

Let $\tau = \mathfrak{m} + \mathfrak{r}_0$ be a standard subalgebra of $\mathfrak{g}$ of nilpotent radical where $\mathfrak{m}$ is its nilpotent part and $\mathfrak{r}_0$ is its semisimple part. The nilpotent part $\mathfrak{m}$ of $\tau$ is a nilpotent standard subalgebra of $\mathfrak{g}$, then $\mathfrak{m}$ is associated to a certain subset $\mathfrak{R}$ of positive roots two by two noncomparable.

We recall that $S_1 = \mathfrak{R} \cap \Pi$, $S_2 = \bigcup\limits_{\omega \in \mathfrak{R}} S^\omega$ and $\Delta_1 = \langle \Pi \backslash S_2 \rangle$. We set $\Omega_1 = \langle \Pi \backslash S_1 \rangle$. We have $\mathfrak{r}_0$ an ideal of the Levi reductive subalgebra $\mathfrak{r}_1 = \mathfrak{h} + \sum\limits_{\alpha \in \Omega_1} \mathfrak{g}_\alpha$ lying in the Levi reductive subalgebra $\mathfrak{r}_2 = \mathfrak{h} + \sum\limits_{\alpha \in \Delta_1} \mathfrak{g}_\alpha$. Since $\mathfrak{r}_0$ is semisimple algebra,

then there exists $\Psi$ a common connected subsystem of $\Pi\backslash S_1$ and $\Pi\backslash S_2$ such that $\mathfrak{r}_o = \sum\limits_{\alpha\in\langle\Psi\rangle} \mathfrak{g}_\alpha + \sum\limits_{\alpha\in\langle\Psi\rangle} \mathbb{C}[e_\alpha, e_{-\alpha}]$.

*Setting the subspace* $P_\Psi^- = \sum\limits_{\alpha\in\langle\Psi\rangle} [\mathfrak{g}_\alpha, \mathfrak{n}_2^-]$ *and the subspace* $P_\Psi^+ = \sum\limits_{\alpha\in\langle\Psi\rangle} [\mathfrak{g}_\alpha, \mathfrak{n}_2^+]$ where $\mathfrak{n}_2^+ = \sum\limits_{\beta\in\Delta_+\backslash\mathfrak{R}_2} \mathfrak{g}_\beta$ and $\mathfrak{n}_2^- = \sum\limits_{\beta\in\Delta_+\backslash\mathfrak{R}_2} \mathfrak{g}_{-\beta}$.

Using the properties of $\mathfrak{m}$ and $\mathfrak{r}_0$, one proves that

$$V_\tau = V_\mathfrak{m} + P_\Psi^- + [h_\Psi, \mathfrak{n}_2^-]$$

with $h_\Psi = \sum\limits_{\alpha\in\langle\Psi\rangle} \mathbb{C}[e_\alpha, e_{-\alpha}]$ .

**Lemma 2.3** *If $\Psi$ is a common connected component of $\Pi\backslash S_1$ and $\Pi\backslash S_2$ then $P_\Psi^+$ is included in $\mathfrak{n}_2^+$.*

**Proof :** Let $\alpha$ be a root such that $\mathfrak{g}_\alpha \subset P_\psi^+$. Then, there exists $\gamma \in \langle\psi\rangle^+$ and $\beta \in \Delta_+\backslash\mathfrak{R}_2$ such that $\alpha = \pm\gamma + \beta \in \Delta_+$. We want to prove that $\mathfrak{g}_\alpha \subset \mathfrak{n}_2^+$.

We assume that $\alpha$ is a root of $\mathfrak{R}_2$. Then, there exists $\omega \in \mathfrak{R}_1$ such that $\pm\gamma + \beta + \omega \in \Delta$. Applying the Jacobi identity to $\pm\gamma$, $\beta$ and $\omega$, we have two cases :
First case : if $\pm\gamma + \omega \in \Delta$, since $\pm\gamma \in \langle\psi\rangle \subset \Delta_1$ then $\pm\gamma + \omega \in \mathfrak{R}_1$. We have $\underbrace{\pm\gamma + \omega}_{\in\mathfrak{R}_1} + \beta$ then $\beta \in \mathfrak{R}_2$. Therefor, we have a contradiction.
Second case : if $\beta + \omega \in \Delta$ then $\beta \in \mathfrak{R}_2$. Therefor, we have a contradiction.

Finally, $\alpha$ is a root of $\Delta_+\backslash\mathfrak{R}_2$ and $\mathfrak{g}_\alpha \subset \mathfrak{n}_2^+$. ∎

**Proposition 2.1** *Let $\alpha$ and $\beta$ be two roots of $\Delta_+\backslash\mathfrak{R}_2$ such that $\alpha + \beta \in \Delta_+$.*
1. *If $\mathfrak{g}_{\alpha+\beta} \subset P_\Psi^+$ then $\mathfrak{g}_\beta \subset P_\Psi^+$ and $\mathfrak{g}_\beta \subset P_\Psi^+$.*
2. *If $\mathfrak{g}_\alpha \subset P_\psi^+$ or $\mathfrak{g}_\beta \subset P_\psi^+$ then $\mathfrak{g}_{\alpha+\beta} \subset P_\psi^+$.*

**Proof :** We have $\alpha + \beta \in \Delta_+\backslash\mathfrak{R}_2$ then, it is necessary that $\alpha \in \langle\Pi\backslash S_1\rangle$ and $\beta \in \langle\Pi\backslash S_1\rangle$ because if no, we have $\alpha$ and $\beta$ are two roots of $\mathfrak{R}_1$. So, by definition of the set $\mathfrak{R}_2$, we have $\alpha$ and $\beta$ in $\mathfrak{R}_2$, contradiction.
1. We have $\mathfrak{g}_{\alpha+\beta} \subset P_\psi^+$ then, there exists $\gamma_0 \in \langle\Psi\rangle^+$ such that $\alpha + \beta \pm \gamma_0 \in \Delta_+\backslash\mathfrak{R}_2$. Applying the Jacobi identity to roots $\alpha$, $\beta$ and $\pm\gamma_0$, we obtain two cases :
First case : if $\alpha \pm \gamma_0 \in \Delta$ then $\alpha \pm \gamma \in \langle\psi\rangle$ because $\psi$ is the connected subsystem of $\Pi\backslash S_1$. Therefore, we have $[\mathfrak{g}_{-\alpha\mp\gamma_0}, \mathfrak{g}_{\beta+\alpha\pm\gamma_0}] = \mathfrak{g}_\beta \subset P_\psi^+$ and $[\mathfrak{g}_{\mp\gamma_0}, \mathfrak{g}_{\alpha\pm\gamma_0}] = \mathfrak{g}_\alpha \subset P_\psi^+$.
Second case : if $\beta\pm\gamma_0 \in \Delta$ then $\beta\pm\gamma_0 \in \langle\psi\rangle$ because $\psi$ is the connected subsystem of $\Pi\backslash S_1$. Therefor, we have $[\mathfrak{g}_{-\beta\mp\gamma_0)}, \mathfrak{g}_{\alpha+\beta\pm\gamma_0}] = \mathfrak{g}_\alpha \subset P_\psi^+$ and $[\mathfrak{g}_{\mp\gamma_0}, \mathfrak{g}_{\beta\pm\gamma_0}] = \mathfrak{g}_\beta \subset P_\psi^+$.

2. Since $\mathfrak{g}_\alpha \subset P_\psi^+$ then, there exists $\alpha_0 \in \Delta\backslash\mathfrak{R}_2$ and $\omega_0 \in \langle\psi\rangle^+$ such that $\alpha = \alpha_0\pm\omega_0$. Applying the Jacobi identity to $\alpha_0$, $\pm\omega_0$ and $\beta$, we have two cases :
First case : if $\alpha_0 + \beta \in \Delta_+\backslash\mathfrak{R}_2$ then $\mathfrak{g}_{\alpha+\beta} \subset P_\psi^+$.
Second case : if $\beta \pm \omega_0 \in \Delta$ then $\beta \pm \omega_0 \in \langle\psi\rangle$ and $\mathfrak{g}_{\alpha+\beta} \subset P_\psi^+$.
If $\mathfrak{g}_\beta \subset P_\psi^+$, we remplace $\alpha$ by $\beta$ and with the same reason we prove that $\mathfrak{g}_{\alpha+\beta} \subset P_\psi^+$.
∎

The next theorem gives the explicit formula of appui subspace $V_\tau$ associated to standard subalgebra $\tau$ of $\mathfrak{g}$.

**Theorem 2.2** *Let $\mathfrak{g}$ be a complex simple Lie algebra and $\tau = \mathfrak{m} + \mathfrak{r}_o$ be a standard subalgebra of $\mathfrak{g}$ of nilpotent radical, where $\mathfrak{m}$ is a nilpotent standard subalgebra associated to a subsystem $\mathfrak{R}$ of positive roots two by two non comparable and $\mathfrak{r}_0$ is an ideal of $\mathfrak{r}_1$ contained in $\mathfrak{r}_2$.*
*Then, we have :*
*1. $V_\tau = V_\mathfrak{m}$ if $P^-_\Psi = \varnothing$.*
*2. $V_\tau = \mathfrak{g}$ if $P^-_\Psi \neq \varnothing$.*

**Proof :**

1. If $P^-_\Psi = \varnothing$ then, $[h_\Psi, \mathfrak{n}^-_2] = \varnothing$. Therefore, $V_\tau = [\tau, \mathfrak{g}] = V_\mathfrak{m}$ and we have the result.

2. If $P^-_\Psi \neq \varnothing$. We have $V_\tau = V_\mathfrak{m} + P^-_\Psi + [h_\Psi, \mathfrak{n}^-_2]$ and $\mathfrak{g} = V_\mathfrak{m} + \mathfrak{n}^-_2$. So, it is enough to show that $P^-_\Psi = \mathfrak{n}^-_2$. It is the same to show that $P^+_\Psi = \mathfrak{n}^+_2$. By using the lemma 2.3, we have $P^+_\Psi \subset \mathfrak{n}^+_2$, then it left to prove that $\mathfrak{n}^+_2 \subset P^+_\Psi$.

We consider $\theta$ the highest root of $\Delta$. In the first, we prove that $\mathfrak{g}_\theta \subset P^+_\psi$.
We have $P^+_\Psi \neq \varnothing$ then, there exists $\alpha$ a root of $\Delta_+ \backslash \mathfrak{R}_2$ such that $\mathfrak{g}_\alpha \subset P^+_\Psi$. Since $\alpha \leqslant \theta$, hence there exists $\theta_1, \cdots, \theta_k$ a positive roots such that $\theta = \alpha + \theta_1 + \cdots + \theta_k$ and $\alpha + \theta_1 + \cdots + \theta_j \in \Delta$ for each $j \in [\![1, k]\!]$. By induction on $k$, we prove that $\mathfrak{g}_\theta \subset P^+_\Psi$.
If $k = 1$, then $\theta = \alpha + \theta_1$ and by applying the proposition 2.1, we have $\mathfrak{g}_\theta \subset P^+_\psi$.
If $k \geqslant 1$. One may write $\theta = \underbrace{\alpha + \theta_1 + \cdots + \theta_{k-1}}_{\theta'} + \theta_k$. By induction $\mathfrak{g}_{\theta'} \subset P^+_\psi$ and by applying the proposition 2.1, we have $\mathfrak{g}_\theta \subset P^+_\psi$.

Now, let $\beta$ be a root such that $\mathfrak{g}_\beta \subset \mathfrak{n}^+_2$. We have $\beta \leqslant \theta$ then there exists $\beta_1, \cdots, \beta_k$ a positive roots such that $\theta = \beta + \beta_1 + \cdots + \beta_s$ and $\beta + \beta_1 + \cdots + \beta_j \in \Delta$ for each $j \in [\![1, s]\!]$. We have $\theta = \underbrace{\beta + \beta_1 + \cdots + \beta_{s-1}}_{\beta'} + \beta_s$, then by applying the proposition 2.1, we have $\mathfrak{g}_{\beta'} \subset P^+_\psi$. Then, we have $\mathfrak{g}_\beta \subset P^+_\psi$. Therefore, we prove that $\mathfrak{n}^+_2 \subset P^+_\psi$ and this implies that $\mathfrak{n}^+_2 = P^+_\psi$ and $\mathfrak{n}^-_2 = P^-_\psi$.

Finally, $[\tau, \mathfrak{g}] = V_\mathfrak{m} + \mathfrak{n}^-_2 = \mathfrak{g}$. So the second result of the theorem is proved. ∎

**Eexample 2.1** *Let $\mathfrak{g}$ be a simple Lie algebra of type $F_4$ and $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a basis of the roots system of $F_4$. We consider the nilpotent standard subalgebra $\mathfrak{m}$ associated to the subsystem $\mathfrak{R} = \{\alpha_3\}$. We have $\Pi \backslash S_2 = \Pi \backslash S_1 = \{\alpha_1, \alpha_2, \alpha_4\}$ and $\Delta_+ \backslash \mathfrak{R}_2 = \{\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\}$.*

1. *If we consider a standard subalgebra defined by $\mathfrak{m}$ and an ideal $\mathfrak{r}_o$ associated to common connected component $\Psi = \{\alpha_4\}$. We have $P^-_\Psi = \varnothing$, in this case $V_\tau = V_\mathfrak{m}$.*

2. *If we consider a standard subalgebra defined by $\mathfrak{m}$ and an ideal $\mathfrak{r}_o$ associated to common connected component $\Psi = \{\alpha_1, \alpha_2\}$. We have $P^-_\Psi \neq \varnothing$, more precisely, $P^-_\Psi = \mathfrak{n}^-_2$. In this case $V_\tau = \mathfrak{g}$.*

# 3 A class of standard subalgebra

In this section, we prove that any standard subalgebra of $\mathfrak{g}(A)$, under certain supplementary condition, is of the form given by theorem 1.1.

Let $\mathfrak{m}$ be a nilpotent standard subalgebra of $\mathfrak{g}$ associated to subsystem $\mathfrak{R} \subset \Delta_+$ of roots two by two non comparable.

**Proposition 3.1** *Let $V_\mathfrak{m}$ be the appui subspace associated to $\mathfrak{m}$ then $[V_\mathfrak{m}, \mathfrak{g}] = \mathfrak{g}$.*

**Proof :** We consider $\mathfrak{n}$ the nilradical part of normalizer of $\mathfrak{m}$. The subalgebra $\mathfrak{n}$ is complete standard subalgebra associated to subsystem $S_2 = \underset{\omega\in\mathfrak{R}}{\cup} S^\omega$. We have $V_\mathfrak{n} = [\mathfrak{n}, \mathfrak{g}] \subset [V_\mathfrak{m}, \mathfrak{g}]$. Since $V_\mathfrak{n} = \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha + \mathfrak{h} + \sum_{\alpha\in\Delta_+\setminus\mathfrak{R}_2^\mathfrak{n}} \mathfrak{g}_{-\alpha}$ with $\mathfrak{R}_2^\mathfrak{n} = \{\alpha \in \Delta_+ : \alpha_\gamma = \theta_\gamma \quad \text{for all } \gamma \in S_2\}$ (remark 2.1) then, it is enough to prove that $\mathfrak{g}_{-\alpha}$ is included in $[V_\mathfrak{m}, \mathfrak{g}]$, for any root $\alpha \in \mathfrak{R}_2^\mathfrak{n}$.
Now, let $\beta$ be the root of $\mathfrak{R}_2^\mathfrak{n}$ then for all $\gamma \in S^\beta$ : $ht_\gamma(\alpha) \geqslant ht_\gamma(\beta)$ where $ht_\gamma(\alpha)$ and $ht_\gamma(\beta)$ is the multiplicity of $\gamma$ in the decomposition of simple roots of $\alpha$ and of $\beta$. Therefore, we have $\beta \leqslant \alpha$ and this implies that $\alpha \in \mathfrak{R}_1$. We deduce that $\mathfrak{g}_{-\alpha} = [[e_\alpha, e_{-\alpha}], \mathfrak{g}_{-\alpha}] \subset [V_\mathfrak{m}, \mathfrak{g}]$. This proves that $V = \mathfrak{g}$. ■

**Remark 3.1** *In proposition 3.1, if we take $\tau$ a standard subalgebra not necessarily nilpotent, we have also $[V_\tau, \mathfrak{g}] = \mathfrak{g}$.*

Let $n$ be a natural number. Let $\{I_{n+j}\}_{j\geqslant 0}$ be a family of vectors subspaces of $\mathfrak{g}$ such that the subspace $\overline{\tau} = \sum_{j\geqslant 0} t^{n+j} \otimes I_{n+j} + \mathbb{C}\mathrm{K}$ is a standard subalgebra of $\mathfrak{g}(A)$ of normalizer $\overline{\rho} = \rho + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathsf{d}$ with $\rho$ is the parabolic subalgebra of $\mathfrak{g}$.

We have $[\overline{\tau}, \overline{\rho}] = t^n \otimes [I_n, \rho] + t^{n+1} \otimes ([I_n, \mathfrak{g}] + [I_{n+1}, \rho]) + t^{n+2} \otimes ([I_n, \mathfrak{g}] + [I_{n+1}, \mathfrak{g}] + [I_{n+2}, \rho]) + t^{n+3} \otimes (\cdots + [I_{n+2}, \mathfrak{g}] + \cdots) + \cdots \subset \overline{\tau}$.

**Lemma 3.1** *We have the following relations :*

1. *$[I_n, \rho] \subset I_n$.*
2. *$[I_n, \mathfrak{g}] \subset I_{n+1}$ and $[I_{n+1}, \rho] \subset I_{n+1}$.*
3. *$[I_{n+1}, \mathfrak{g}] \subset I_{n+2}$ and $[I_{n+2}, \rho] \subset I_{n+2}$.*
4. *For all $j \geqslant 3$ : $[I_{n+2}, \mathfrak{g}] \subset I_{n+j}$.*

**Proposition 3.2** *If $I_n$ is a Lie subalgebra of $\mathfrak{g}$ then $I_n$ is included in $\rho$.*

**Proof :** We can write $\rho = \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha + \mathfrak{h} + \sum_{\alpha\in\langle\Pi\setminus T\rangle^+} \mathfrak{g}_{-\alpha}$, for a certain subset $T$ of $\Pi$.
We suppose that $I_n$ is not in $\rho$ and we set $T_1 = \{\beta \in T,\ \mathfrak{g}_{-\beta} \subset I_n\}$. We consider $\rho_1 = \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha + \mathfrak{h} + \sum_{\alpha\in\langle\Pi\setminus(T\setminus T_1)\rangle^+} \mathfrak{g}_{-\alpha}$ the parabolic subalgebra of $\mathfrak{g}$ associated to $(T\setminus T_1)$.
By definition $\rho$ is included in $\rho_1$.

We want to prove that $\overline{\tau}$ is an ideal of $\rho_1 + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathsf{d}$.
First, we prove that $I_n$ is an ideal of $\rho_1$. Let $\alpha$ and $\beta$ be two roots such that $\mathfrak{g}_\alpha \subset I_n$, $\mathfrak{g}_\beta \subset \rho_1$ and $\alpha + \beta$ is the root.
If $\beta$ is a root of $\Delta_+ \cup \langle\Pi\setminus T\rangle^-$ then $\mathfrak{g}_\beta$ is included in $\rho$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset [I_n, \rho] \subset I_n$.
If $\beta$ is a root of $\langle\Pi\setminus(T\setminus T_1)\rangle^-$ with $C_\beta \cap T_1 \neq \varnothing$. There exists $\beta_1, \cdots, \beta_r$ a simple

roots such that $\beta = -\beta_1 - \cdots - \beta_r$ and for all $j \in [\![1, r]\!]$, $\beta_1 + \cdots + \beta_j \in \Delta$. Let $i$ be the smallest index such that $\beta_i \in T_1$. We have $\beta'_{i-1} = \beta_1 + \cdots + \beta_i \in \langle \Pi \backslash T \rangle$ and $\mathfrak{g}_{-\beta'_i} = [\mathfrak{g}_{-\beta_i}, \mathfrak{g}_{-\beta'_{i-1}}] \subset [I_n, \rho] \subset I_n$. Now taking the simple root $\beta_{i+1}$, we have two cases :

If $\beta_{i+1} \in T_1$, we have $\mathfrak{g}_{-\beta'_{i+1}} = [\mathfrak{g}_{-\beta'_i}, \mathfrak{g}_{-\beta_{i+1}}] \subset [I_n, I_n] \subset I_n$ (because $I_n$ is subalgebra).

If $\beta_{i+1} \in \Pi \backslash T$, we have $\mathfrak{g}_{-\beta'_{i+1}} = [\mathfrak{g}_{-\beta_{i+1}}, \mathfrak{g}_{-\beta'_i}] \subset [\rho, I_n] \subset I_n$.
With similar argument, we show that $\mathfrak{g}_\beta \subset I_n$.

Therefore, $\mathfrak{g}_{\alpha+\beta} = [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset [I_n, I_n] \subset I_n$. Then, $I_n$ is the standard subalgebra of $\mathfrak{g}$ of normalizer $\rho_1$.

Now, we have $V_{I_n} = [I_n, \mathfrak{g}] \subset I_{n+1}$ and $[I_{n+1}, \rho_1] \subset [I_{n+1}, \rho] + [I_{n+1}, I_n] \subset I_{n+1}$.

Since $I_n$ is a standard subalgebra then, by applying the proposition 3.1 an the relation 3 of the lamma 3.1, we have $\mathfrak{g} = [V_{I_n}, \mathfrak{g}] \subset [I_{n+1}, \mathfrak{g}] \subset I_{n+2}$. Then $I_{n+2} = \mathfrak{g}$. Since $\mathfrak{g}$ is the simple Lie algebra, for $j \geqslant 3$, we have $[I_{n+2}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g} \subset I_{n+j}$. This implies that $I_{n+j} = \mathfrak{g}$.

Finally, we have proved that $\overline{\tau}$ is an ideal of the parabolic subalgebra $\rho_1 + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathfrak{d}$ and this subalgebra contains the normalizer $\rho + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathfrak{d}$ of $\overline{\tau}$, contradiction. Hence $I_n$ is included in $\rho$. ∎

**Remark 3.2** *The proposition 3.2 proves that $I_n$ is the standard subalgebra of normalizer $\rho$.*

The following theorem determines a class of standard subalgebra of $\mathfrak{g}(A)$ under certain condition.

**Theorem 3.1** *Let $\overline{\tau} = \sum_{j \geqslant 0} t^{n+j} \otimes I_{n+j} + \mathbb{C}\mathrm{K}$ be a standard subalgebra of $\mathfrak{g}(A)$ of normalizer $\overline{\rho} = \rho + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathfrak{d}$ and $I_n \neq 0$.*

*If $I_n$ is a Lie subalgebra of $\mathfrak{g}$ then $\overline{\tau} = t^n \otimes \tau + t^{n+1} \otimes V + t^{n+2}\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K}$ where $\tau = I_n$ is a standard subalgebra of $\mathfrak{g}$ of normalizer $\rho$, $V$ is a subspace of $\mathfrak{g}$ contain the subspace $V_\tau = [\tau, \mathfrak{g}]$ and $I_{n+j} = \mathfrak{g}$, for all $j \geqslant 2$.*

**Proof :** Let $n$ be a natural number. We set $I_n = \tau$. We write $\overline{\tau} = t^n \otimes \tau + \sum_{j \geqslant 1} t^{n+j} \otimes I_{n+j} + \mathbb{C}\mathrm{K}$.

By applying the proposition 3.2, we have $\tau$ is included in $\rho$. Then, $\tau$ is the standard subalgebra of $\mathfrak{g}$ of normalizer $\rho$.

We set $V = I_{n+1}$. The relation 2. in the lemma 3.1 proves that $V_\tau$ is included in $V$ where $V_\tau$ is the appui subspace associated to $\tau$.

We prove by induction on $j \geqslant 2$ that $I_{n+j} = \mathfrak{g}$. We use the relation 3 in the lemma 3.1, then we have $[V_\tau, \mathfrak{g}] \subset I_{n+2}$. By applying the proposition 3.1 and remark 3.1, we have $I_{n+2} = \mathfrak{g}$.

Since $\mathfrak{g}$ is the simple Lie algebra, then we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Using the relation 4 in the lemma 3.1, we have $I_{n+j} = \mathfrak{g}$. Therefore $\overline{\tau} = t^n \otimes \tau + t^{n+1} \otimes V + t^{n+2}\mathbb{C}[t] \otimes \mathfrak{g}$. ∎

**Corollary 3.1** *Let $\overline{\tau} = I_0 + \sum_{k \geqslant 1} t^k \otimes I_k + \mathbb{C}\mathrm{K} + \mathbb{C}\mathfrak{d}$ be the standard subalgebra of $\mathfrak{g}(A)$ of normalizer $\rho(\overline{\tau}) = \rho + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathfrak{d}$.*

*Then $\tau = I_0$ is a standard subalgebra of $\mathfrak{g}$ of normalizer $\rho$ and $\overline{\tau} = \tau + t\mathbb{C}[t] \otimes \mathfrak{g} + \mathbb{C}\mathrm{K} + \mathbb{C}\mathfrak{d}$.*

**Proof :** It is an immediate consequence of the previous theorem. ■

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